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The perturbed ladder operator method. Perturbed eigenvalues and eigenfunctions from finite difference considerations

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Abstract. The finite difference aspect of the perturbed ladder operator method is reinvestigated. By the use of finite difference calculus, resolution of the factorisability condition is achieved, at any order of perturbation, without assuming for the ladder and factorisation functions any particular dependence on the quantum number. A novel procedure of obtaining perturbed eigenfunctions in terms of the unperturbed functions is described. The method, which holds for any type of factorisation (types A to F), is applied to resolution of the type A wave equation with potential

$$V(x)=-a^2\left(\frac{m^2-\frac{1}{4}+d^2+2dm\cos a(x+p)}{\sin^2 a(x+p)}+\sum_s b_s[\cos a(x+p)]^s\right).$$

The perturbed type A problem, which has not been previously treated, contains, as particular cases, the perturbation of the spherical harmonics Y_l^m (or generalised $Y_{l,\gamma}^m$) functions, of the symmetric top functions $\mathscr{D}^j_{mk}$ and, more generally, of the hypergeometric functions $F(\alpha, \beta; \gamma; x)$.

1. Introduction

The main features of the perturbed ladder operator method have been described in previous papers (Bessis *et al* 1978, 1980, to be referred to as I and II). This method, which maps the perturbation scheme onto the exact ladder operator formalism, enables one to treat non-factorisable Sturm–Liouville eigenequations in the same way as factorisable ones and enlarges the field of application of the original Schrödinger, Infeld and Hull factorisation method (Schrödinger 1940, 1941, Infeld and Hull 1951). From the computational point of view, the critical part of the perturbed ladder scheme is the determination of perturbed ladder operators and associated factorisation functions. Once the ladder and factorisation functions are determined, one obtains closed form expressions for the perturbed eigenfunctions and eigenvalues in terms of the quantum numbers of the unperturbed problem without having to calculate any matrix elements (see papers I and II). The unperturbed kernel potential has to be one of the six Infeld–Hull exact factorisation types (types A to F). In fact these six types are interrelated (Infeld and Hull 1951, Bessis *et al* unpublished) and, by an adequate transformation of variable and function, they can be ultimately reduced to two fundamental types we shall hereafter distinguish as 'radial' types (types B, C, D and F) and 'trigonometric' types (types A and E). In papers I and II, particular attention has been paid to the perturbed 'radial' types and solutions for these types have been found in the case where the required factorising potentials are assumed to involve only positive powers of the quantum number. Nevertheless, when the same procedure is applied to the 'trigonometric' types it leads to intricate calculations and to perturbed factorising potentials of restricted physical interest.

In the present paper, the finite difference aspect of the perturbed ladder operator method is re-examined. By the use of finite-difference calculus, the solution of the factorisability condition is achieved, at any order of the perturbation, without assuming for the ladder and factorisation functions any particular dependence on the quantum number. Closed-form expressions of the associated perturbed eigenvalues and perturbed 'key' eigenfunctions are given. Thus, starting from the perturbed 'key' function, one can obtain, stepwise, either by the use of the ladder operator or by the use of the equivalent three-term recurrence relation (see paper II), closed-form expressions of any perturbed eigenfunction. The procedure is straightforward but becomes somewhat tedious far from the 'key'. However, by use of the perturbed ladder function, an alternative procedure which leads to closed-form expressions of the perturbed eigenfunctions in terms of the unperturbed functions is obtained. This is particularly recommended when eigenfunctions far from the 'key' are needed. All these results, which hold for any type of factorisation (types A to F), are described in §2. Particular attention is focused on type A (§3) which concerns, *ex generalis*, the perturbation of spherical harmonics, symmetric top and hypergeometric $F(\alpha, \beta; \gamma; x)$ functions. Because of the existence of interrelation between types of factorisation, one can obtain many recurrence relations amongst the eigenfunctions of a given factorisable equation. Only those which are required for our calculations are given below. It should be noted that these include, as a particular case, recurrence relations between Gegenbauer and Jacobi polynomials which have been previously derived by Miller from group theory considerations (Miller 1968). As an illustrative example, our procedure is applied, up to the second order of the perturbation, to the resolution of the perturbed type A eigenequation (§4).

2. The perturbed factorisation scheme

It is almost unavoidable to recall briefly the exact factorisation scheme and the fundamental factorisability conditions, since they are of continual use throughout this paper.

2.1. *Exact factorisation*

Let us consider a second-order differential eigenequation which has been reduced to the standard form

$$\left(\frac{d^2}{dx^2} + U(x, m) + \Lambda_j\right) \psi_{jm} = 0 \tag{1}$$

associated with the boundary conditions $(x_1 \leqslant x \leqslant x_2)$

$$|\psi(x_1)|^2 = |\psi(x_2)|^2 = 0 \qquad \int_{x_1}^{x_2} |\psi(x)|^2 \, dx = 1 \tag{2}$$

where $m = m_0, m_0+1, m_0+2, \ldots$ is a quantum number which takes successive discrete values labelling the eigenfunctions.

Such an equation (1) is factorisable when it can be replaced by each of the following two difference differential equations

$$\begin{aligned}\left(K(x, m+1)+\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(K(x, m+1)-\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm} &= (\Lambda_j - L(m+1))\psi_{jm}\\ \left(K(x, m)-\frac{\mathrm{d}}{\mathrm{d}x}\right)\left(K(x, m)+\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm} &= (\Lambda_j - L(m))\psi_{jm}\end{aligned} \tag{3}$$

where j is the quantum number associated with the eigenvalues Λ_j, $K(x, m)$ is the ladder function and $L(m)$ is the factorisation function which does not depend on x. The necessary condition for the existence of a quadratically integrable solution of equation (1), i.e. the quantification condition, is $\varepsilon(j-|m|) = v = \text{integer} \geqslant 0$ where $\varepsilon = 1$ (or $\varepsilon = -1$) according to whether $L(m)$ is an increasing (or decreasing) function of m. The associated eigenvalues are

$$\Lambda_j = L(j+\tfrac{1}{2}+\tfrac{1}{2}\varepsilon). \tag{4}$$

The normalised eigenfunctions ψ_{jm} are solutions of the following pair of difference differential equations

$$\begin{aligned}\left(K(x, m)+\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm} &= N_j(m)\psi_{j,m-1}\\ \left(K(x, m+1)-\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm} &= N_j(m+1)\psi_{j,m+1}\end{aligned} \tag{5}$$

with

$$N_j(m) = (\Lambda_j - L(m))^{1/2}.$$

These equations allow the determination of any $\psi_{jm}(x)$ function from the knowledge of the 'key' function $\psi_{jj}(x)$ which is a solution of the first-order differential equation

$$\left(K(x, j+\tfrac{1}{2}+\tfrac{1}{2}\varepsilon)-\varepsilon\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jj} = 0. \tag{6}$$

Moreover, it follows immediately from (5) that the ψ_{jm} functions satisfy the following three-term (non-differential) recurrence relation

$$(K(x, m)+K(x, m+1))\psi_{j,m}(x) = N_j(m)\psi_{j,m-1}(x)+N_j(m+1)\psi_{j,m+1}(x). \tag{7}$$

As is well known (Infeld and Hull 1951), there are six fundamental types (denoted types A to F) of potential functions $U^{(0)}(x, m)$ with associated $K^{(0)}(x, m)$ and $L^{(0)}(m)$ allowing direct factorisation of eigenequation (1) with $U(x, m) = U^{(0)}(x, m)$ and leading to eigenfunctions $\psi_{jm}{}^{(0)}$ which involve classical orthogonal polynomials (Hadinger *et al* 1974). For types A and E, which are of interest to us in the present paper, results are collected in table 1. As pointed out by Infeld and Hull, the factorisation scheme may also be extended to solve eigenequation (1) in the case of a potential function $V(x, m)$ which differs from U only in its dependence on m. In order to identify $V(x, m)$ with $U(x, m)$, one considers the $U(x, m)$ potential as 'embedded' in a potential function $u(x, m, \mu)$ which depends on a supplementary 'artificial' parameter μ such that $u(x, m, \mu)$ can be identified in m with $V(x, m)$ and that $u(x, m, \mu = \varphi(m)) \equiv U(x, m)$.

Table 1. Trigonometric Infeld–Hull exact factorisation types.

	Type A	Type E
$U^{(0)}(x, m)$	$\dfrac{-a^2(m^2-\frac{1}{4}+d^2+2md\cos a(x+p))}{\sin a(x+p)}$	$\dfrac{-a^2m(m+1)}{\sin^2 a(x+p)}-2\,aq\cot a(x+p)$
$K^{(0)}(x, m)$	$a\,m\cot a(x+p)+\dfrac{ad}{\sin a(x+p)}$	$am\cot a(x+p)+q/m$
$L^{(0)}(m)$	$a^2(m-\frac{1}{2})^2$	$a^2m^2-q^2/m^2$
$\psi^{(0)}_{jm}(x)$	$C_j(m)(\sin\frac{1}{2}a(x+p))^{\alpha+1/2}(\cos\frac{1}{2}a(x+p))^{\beta+1/2}P_v^{(\alpha,\beta)}[\cos\frac{1}{2}a(x+p)]$	$C_j(m)\,(\sin a(x+p))^{\varepsilon(j+1/2)+1/2}\exp\left(\dfrac{\varepsilon q(x+p)}{(j+\frac{1}{2}+\frac{1}{2}\varepsilon)}\right)P_v^{(\alpha,\beta)}(\coth \mathrm{i}a(x+p))$
	with	with
	$v=\varepsilon(j-\lvert m\rvert)=\text{integer}\geqslant 0$	$v=\varepsilon(j-\lvert m\rvert)=\text{integer } 7, 0$
	$\alpha=\varepsilon(m+d)$	$\alpha=-\varepsilon[(j+\frac{1}{2}+\frac{1}{2}\varepsilon)-\mathrm{i}q/a(j+\frac{1}{2}+\frac{1}{2}\varepsilon)]$
	$\beta=\varepsilon(m-d)$	$\beta=-\varepsilon[(j+\frac{1}{2}+\frac{1}{2}\varepsilon)+\mathrm{i}q/a(j+\frac{1}{2}+\frac{1}{2}\varepsilon)]$

$P_v^{(\alpha,\beta)}$ is the Jacobi polynomial.

Thus the ladder and factorisation functions associated with $u(x, m, \mu)$ both depend on the parameter μ and lead to μ-parametrised eigenvalues and eigenfunctions. At the end of the factorisation scheme one merely sets $\mu = \varphi(m)$ to obtain the required eigenvalues

$$\Lambda_{jm} = L(j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\,; \mu = \varphi(m))$$

and the required eigenfunctions $\psi_{jm}(x, \mu = \varphi(m))$. As will be seen in the following sections, this 'artificial' or 'embedded' factorisation procedure is very useful in the 'perturbed ladder' scheme.

2.2. Factorisability condition of the perturbed eigenequation

In the previous papers I and II, it has been shown that the original range of applicability of the exact factorisation method can be extended to the solution of eigenequation (1) with a potential function $U(x, m)$, which can be expanded in a perturbation series in the parameter η such that

$$U(x, m) = U^{(0)}(x, m) + \eta U^{(1)}(x, m) + \eta^2 U^{(2)}(x, m) + \cdots + \eta^N U^{(N)}(x, m) \tag{8}$$

where $U^{(0)}(x, m)$ is one of the six Infeld–Hull factorisation types.

The procedure amounts to finding associated ladder and factorisation functions

$$\begin{aligned} K(x, m) &= K^{(0)}(x, m) + \eta K^{(1)}(x, m) + \eta^2 K^{(2)}(x, m) + \cdots + \eta^N K^{(N)}(x, m) \\ L(m) &= L^{(0)}(m) + \eta L^{(1)}(m) + \eta^2 L^{(2)}(m) + \cdots + \eta^N L^{(N)}(m) \end{aligned} \tag{9}$$

in such a way as to satisfy (3) up to a given power N of the parameter η. Once the perturbed $U(x, m)$, $K(x, m)$ and $L(m)$ functions are known, the perturbed problem (up to the Nth order) may be handled in the same way as the exact factorisable (unperturbed) problem. Of course, specific expressions for the perturbed ladder and factorisation functions correspond to each type of factorisation associated with the unperturbed potential.

From the comparison of equations (1) and (3), it is easily shown that the necessary and sufficient condition to be fulfilled by the required $U^{(N)}(x, m)$, $K^{(N)}(x, m)$ and $L^{(N)}(m)$ is

$$\begin{aligned} &\sum_{\nu=0}^{N} K^{(\nu)}(x, m+1) K^{(N-\nu)}(x, m+1) + \frac{\mathrm{d}}{\mathrm{d}x} K^{(N)}(x, m+1) + L^{(N)}(m+1) = -U^{(N)}(x, m) \\ &\sum_{\nu=0}^{N} K^{(\nu)}(x, m) K^{(N-\nu)}(x, m) - \frac{\mathrm{d}}{\mathrm{d}x} K^{(N)}(x, m) + L^{(N)}(m) = -U^{(N)}(x, m). \end{aligned} \tag{10}$$

These equations are solved recursively; i.e. when considering the determination of $U^{(N)}$, $K^{(N)}$ and $L^{(N)}$, it is assumed that all the $K^{(\nu)}$ for $\nu = 1, 2, \ldots, N-1$ have already been found. The finite difference aspect of equation (10) determines the m dependence of the functions while its differential aspect determines their x dependence.

2.3. Finite difference treatment of the factorisability condition

In the present paper, the solution of the factorisability equations (10) is re-examined without introducing any restricting condition on the m dependence of the $K^{(N)}(x, m)$, $L^{(N)}(m)$ and $U^{(N)}(x, m)$ functions. In order to use the finite difference calculus, it is convenient to introduce the usual first difference Δ and mean M operators for

$$\Delta F(m)=F(m+1)-F(m) \qquad MF(m)=\tfrac{1}{2}(F(m+1)+F(m)). \tag{11}$$

Then, the difference differential equations (10) can be rewritten

$$2\Delta(K^{(0)}(x,m)K^{(N)}(x,m))+2M\frac{\mathrm{d}}{\mathrm{d}x}K^{(N)}(x,m)$$
$$=-\Delta\Big(L^{(N)}(m)+\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m)\Big) \tag{12a}$$

$$U^{(N)}(x,m)=\Big(\frac{\mathrm{d}}{\mathrm{d}x}-2K^{(0)}(x,m)\Big)K^{(N)}(x,m)-L^{(N)}(m)-\sum_{\nu=1}^{N-1}K^{(\nu)}(x,m)K^{(N-\nu)}(x,m). \tag{12b}$$

The first equation (12a) is used to determine the ladder and factorisation functions $K^{(N)}(x, m)$ and $L^{(N)}(m)$. Once they are known the required potential functions $U^{(N)}(x, m)$ are given by (12b).

Let us first consider the x dependence of equation (12a) and assume that the perturbed ladder function can be written

$$K^{(N)}(x,m)=\kappa(x)\sum_{v=0}^{S_N}\gamma_{v+1}{}^{(N)}(m)(y(x))^{v} \tag{13}$$

where, for each type of factorisation, the $\kappa(x)$ and $y(x)$ functions have to be found such that both sides of equation (12a) can be identified on the finite basis of the $(y(x))^{v}$. A sufficient condition is that $\mathrm{d}K^{(N)}/\mathrm{d}x$ as well as the products $K^{(0)}K^{(N)}$ and $K^{(\nu)}K^{(N-\nu)}$ can be expanded as polynomials in the new variable $y(x)$. When this condition has been fulfilled, by equating the coefficients of y^{v} in both sides of equation (12a), one obtains recursive finite difference equations allowing the determination of the $\gamma_v^{(N)}(m)$ and of $L^{(N)}(m)$. Since these finite difference equations are of the first order, each solution $\gamma_v^{(N)}$ ($v=1, S_N+1$) (or $L^{(N)}(m)$) involves one arbitrary constant denoted $k_v^{(N)}$ (or $k_0^{(N)}$ for $L^{(N)}$). The associated potential function $U^{(N)}(x, m)$ is given by (12b) and, through the $K^{(N)}$ and $L^{(N)}$ functions, involves the free constants, $k_v^{(\nu)}$, to be adjusted in order to match the total factorising potential, $U(x, m)$, with a given physical potential function, $V(x, m)$.

Owing to the above condition on the x dependence of the functions $K^{(\nu)}$, it is easily seen that $U^{(N)}(x, m)$ is a finite series of powers of $y(x)$. Consequently, the perturbed ladder operator method can be applied to physical problems leading to the solution of a wave equation (1) with a potential function $V(x, m)$ which can be written

$$V(x,m)=U^{(0)}(x,m)+\eta V^{(1)}(x)+\eta^2V^{(2)}(x)+\ldots+\eta^N V^{(N)}(x) \tag{14}$$

where $U^{(0)}(x, m)$ is one of the six Infeld–Hull factorisation types and the $V^{(N)}(x)$ are polynomial in $y(x)$

$$V^{(N)}(x)=\sum_{v}b_v^{(N)}(y(x))^{v}. \tag{15}$$

In most problems of physical interest, the $b_v^{(N)}$ constants, which are specific to the physical problem under consideration, will not involve m†. Consequently, in order to match $V(x, m)$ with the factorising potential $U(x, m)$, one has to resort to the 'embedded' factorisation. This is conveniently achieved by considering the following embedding potential function with artificial parameter μ

$$u(x, m, \mu) = U^{(0)}(x, m) + \eta U^{(1)}(x, m=\mu) + \ldots + \eta^N U^{(N)}(x, m=\mu). \tag{16}$$

Then, one can determine the set $(k^{(N)})$ of the arbitrary constants, $k_v^{(\nu)}$ $(\nu = 1, \ldots N; v = 0, \ldots S_\nu + 1)$, in terms of the set, $(b^{(N)})$, of the $b_v^{(\nu)}$ from the following identification

$$U^{(N)}(x, m=\mu; k^{(N)}) = V^{(N)}(x; b^{(N)}). \tag{17}$$

In fact, since $U^{(N)}(x, m)$ is given by equation (12*b*), this can be achieved without explicit calculation of the $U^{(N)}(x, m)$ functions. Setting $m = \mu$ in (12*b*), one uses the following relation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 2K^{(0)}\right) K^{(N)} - L^{(N)} - \sum_{\nu=1}^{N-1} K^{(\nu)} K^{(N-\nu)} = V^{(N)}(x; b^{(N)}) \tag{18}$$

where the shortened notation $K^{(0)} = K^{(0)}(x, m=\mu)$, $K^{(N)} = K^{(N)}(x, m=\mu;\, k^{(N)})$ and $L^{(N)} = L^{(N)}(m=\mu;\, k^{(N)})$ has been used.

Once the set, $(k^{(N)})$, of the free constants of the $\gamma_v^{(\nu)}(m)$ coefficients and of the $L^{(\nu)}$ have been expressed in terms of μ and of the set $(b^{(N)})$, one obtains the ladder and factorisation functions associated with the physical problem under consideration

$$K^{(N)}(x, m; \mu; b^{(N)}) = \kappa(x) \sum_{v=0}^{S_N} \gamma_{v+1}^{(N)}(m; \mu; b^{(N)})(y(x))^v \tag{19}$$

$$L^{(N)}(m; \mu; b^{(N)}) = \left(-2K^{(0)}(x, m)K^{(N)} - \sum_{\nu=1}^{N-1} K^{(\nu)} K^{(N-\nu)} - 2\Delta^{-1} M \frac{\mathrm{d}}{\mathrm{d}x} K^{(N)}\right)_{y=0} \tag{20}$$

where $K^{(\nu)}$ is given by (19) as a function of x, m, μ and of the parameters $b_v^{(\nu)}$ of the physical problem under consideration.

Now, one can apply (up to the Nth order of the perturbation) the usual exact factorisation scheme (equations (2) to (7)) with the total ladder and factorisation functions (9) where the $K^{(\nu)}$ and $L^{(\nu)}$ are given by (19) and (20).

2.4. Closed-form expressions of the perturbed eigenvalues and eigenfunctions

The perturbed eigenvalue is (see equation (4))

$$\Lambda_{jm} = L^{(0)}(j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) + \sum_{\nu=1}^{N} L^{(\nu)}(m = j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon;\, \mu = m;\, b^{(N)}) \tag{21}$$

where $\varepsilon = +1$ (or $\varepsilon = -1$) according as the unperturbed factorisation function $L^{(0)}(m)$ is an increasing (or decreasing) function of m.

The normalised 'key' perturbed eigenfunction ψ_{jj} is the solution of the first-order differential equation (6). One gets in terms of the unperturbed normalised key function $\psi_{jj}^{(0)}$

$$\psi_{jj} = \psi_{jj}^{(0)} \exp\left(\varepsilon \sum_{\nu=1}^{N} \eta^\nu \int K^{(\nu)}(x, j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon;\, \mu;\, b^{(\nu)})\, \mathrm{d}x\right). \tag{22}$$

† If $V^{(N)}(x)$ would involve m such that it cannot be identified in m with $U^{(N)}(x, m)$, one could introduce another additional artificial parameter μ' then use $V^{(N)}(x, m = \mu') = \sum_v b_v^{(N)}(\mu')(y(x))^v$ and set $\mu' = m$ at the final stage.

The closed-form expression of any normalised perturbed ψ_{jm} function can be obtained in terms of the factorisation instruments and of the key function ψ_{jj} by use of the recurrence relation (7) (see paper II).

In view of further applications, it may be useful to have at our disposal closed-form expressions of the normalised perturbed eigenfunctions in terms of the unperturbed functions. As will be shown in the following section such an expression can be easily derived once the products $K^{(\nu)}\psi_{jm}^{(0)}$ have been expanded in terms of the unperturbed functions.

Up to now the above results are general and concern all types (A, F). Let us focus our attention on the trigonometric type A which has not been previously treated. It should be mentioned that, owing to the interrelation between types of factorisation (see appendix 1), perturbed factorisation results for type A can be used for type E.

3. Type A perturbed factorisation

As pointed out before, the solution of the perturbed eigenequation is performed in three stages. First one determines, up to the Nth order of the perturbation, the perturbed ladder and factorisation functions in terms of the free constants $k_v^{(\nu)}$. Then, the set of these free constants is related to the set of the parameters $b_v^{(\nu)}$ of the potential under consideration. Finally, one obtains (up to the Nth order of the perturbation) the perturbed eigenvalues and the perturbed eigenfunctions either by use of the recurrence relation (7) or in terms of the unperturbed functions.

3.1. Perturbed ladder and factorisation functions

Owing to the expression of $K^{(0)}(x, m)$ (see table 1), it is found that a convenient expression of $K^{(N)}(x, m)$ which satisfies the above mentioned conditions is

$$K^{(N)}(x, m) = a \sin a(x+p) \sum_{v=0}^{S_N} \gamma_{v+1}^{(N)}(m)\,[\cos a(x+p)]^v. \tag{23}$$

Making use of this expression and of equation (12a), then equating the coefficients of $y^v = [\cos a(x+p)]^v$, for $1 \leqslant v \leqslant S_N + 1$, in both sides of equation (12a), one gets the following finite difference equation allowing the determination of the $\gamma_v^{(N)}(m)$

$$\begin{aligned}(2m+1+v)\gamma_v^{(N)}(m+1) - (2m-1-v)\gamma_v^{(N)}(m)\\ = -2d\Delta\gamma_{v+1}^{(N)}(m) + 2(v+1)M\gamma_{v+2}^{(N)}(m) - \Delta w_v^{(N)}(m)\end{aligned} \tag{24}$$

where the $w_v^{(N)}(m)$ originate from the preceding orders of the perturbation and are defined by

$$\sum_{\nu=1}^{N-1} K^{(\nu)}(x, m)K^{(N-\nu)}(x, m) = a^2 \sum_{v=0} w_v^{(N)}(m)\,[\cos a(x+p)]^v. \tag{25}$$

The upper limit of v on the right-hand side of (25) is the maximum value of $(S_\nu + S_{N-\nu} + 2)$.

It should be noted that while at first order $N = 1$, the upper bound S_1 involved in $K^{(1)}(x, m)$ can be arbitrarily chosen, this is not true for the other S_N. Indeed, owing to the value of the upper bound in (25), it is easily inferred that S_N must satisfy the following condition:

$$S_N \geqslant \max(S_\nu + S_{N-\nu} + 1). \tag{26}$$

Starting from $v = S_N + 1$, the finite difference equation (24) is solved recursively, reducing the integer v stepwise down to 1. Thus, when tackling the determination of $\gamma_v^{(N)}$, the $\gamma_{v+1}^{(N)}$ and $\gamma_{v+2}^{(N)}$ are already known. After multiplying both sides of (24) by $(2m+v-1)!!/(2m-v-1)!!$ one obtains the following complete first-order linear difference equation

$$\Delta \frac{(2m+v-1)!!}{(2m-v-3)!!} \gamma_v^{(N)}(m) = \frac{(2m+v-1)!!}{(2m-v-1)!!} [-\Delta(w_v^{(N)}(m) + 2d\gamma_{v+1}^{(N)}(m)) + 2(1+v)M\gamma_{v+2}^{(N)}(m)] \tag{27}$$

where $(2n)!! = (2n)(2n-2)\ldots 6\times 4\times 2$ and $(2n+1)!! = (2n+1)(2n-1)\ldots 5\times 3\times 1$. By symbolic multiplication with Δ^{-1} (Jordan 1965) one obtains

$$\gamma_v^{(N)}(m) = -\frac{1}{(2m-v-1)}(w_v^{(N)}(m) + 2d\gamma_{v+1}^{(N)}(m) + (v+1)\gamma_{v+2}^{(N)}(m)) + \frac{(2m-v-3)!!}{(2m+v-1)!!}(\Delta^{-1} f_v^{(N)}(m+1) + k_v^{(N)}) \tag{28}$$

where $k_v^{(N)}$ is an arbitrary constant which does not depend on m, and

$$f_v^{(N)}(m) = \frac{2(2m+v-3)!!}{(2m-v-1)!!}(vw_v^{(N)}(m) + 2dv\gamma_{v+1}^{(N)}(m) + (2m-1)(v+1)\gamma_{v+2}^{(N)}(m)). \tag{29}$$

From (25)

$$w_v^{(N)}(m) = \sum_{\nu=1}^{N-1} \sum_{t=t_0}^{t_M} \gamma_t^{(\nu)} (\gamma_{v+2-t}^{(N-\nu)} - \gamma_{v-t}^{(N-\nu)}) \tag{30}$$

with $t_0 = \max(1, v-1-S_{N-\nu})$ and $t_M = \min(S_\nu + 1, v+1)$.

At the first order $N = 1$ of the perturbation, since $w_N^{(1)}(m) \equiv 0$, the starting function of the recursive solution of (28) is

$$\gamma_{S_1+1}^{(1)} = \frac{(2m - S_1 - 4)!!}{(2m + S_1)!!} k_{S_1+1}^{(1)}. \tag{31}$$

When equating the x-independent terms in (12a) one obtains

$$\Delta L^{(N)}(m) = -a^2(2d\Delta\gamma_1^{(N)}(m) - 2M\gamma_2^{(N)}(m) + \Delta w_0^{(N)}(m)). \tag{32}$$

By symbolic multiplication with Δ^{-1}, one obtains

$$L^{(N)}(m) = -a^2(w_0^{(N)}(m) + 2d\gamma_1^{(N)}(m) + \gamma_2^{(N)}(m) - 2\Delta^{-1}\gamma_2^{(N)}(m+1) + k_0^{(N)}) \tag{33}$$

where $k_0^{(N)}$ is an arbitrary constant which does not depend on m.

Keeping in mind that $\Delta^{-1}\gamma_2^{(N)}(m+1)$ has already been calculated in order to obtain $\gamma_1^{(N)}(m)$, the determination of $L^{(N)}(m)$ does not need any further 'finite integration'.

Once the 'factorisation instruments' $K^{(N)}(x, m)$ and $L^{(N)}(m)$, involving the arbitrary constants $k_v^{(\nu)}$ $(\nu = 1, N; v = 0, S_\nu + 1)$, have been obtained, one can apply the perturbed factorisation scheme to the solution of the following eigenequation:

$$\left(\frac{d^2}{dx^2}-\frac{a^2[(m-\frac{1}{2})(m+\frac{1}{2})+d^2+2m\,d\cos a(x+p)]}{\sin^2 a(x+p)}+\sum_{\nu=1}^{N} V^{(\nu)}(x)+\Lambda_{jm}\right)\psi_{jm}=0 \tag{34}$$

with

$$V^{(\nu)}(x)=-a^2\sum_{v=1}^{S_\nu+1} b_v^{(\nu)}\,[\cos a(x+p)]^v. \tag{35}$$

In order to express the total factorisation instruments $K(x, m)$ and $L(m)$ as well as the factorising potential $U(x, m)$ in terms of the physical data, one has to determine the set $(k^{(N)})$ of the free constants in terms of the set $(b^{(N)})$. Making use of expressions (23), (25) and (35) together with equation (18), and equating the coefficients of $[\cos a(x+p)]^v$ in both sides of (18), one can write

$$(2\mu-1-v)\gamma_v^{(N)}(\mu)+2d\gamma_{v+1}^{(N)}(\mu)+(v+1)\gamma_{v+2}^{(N)}(\mu)+w_v^{(N)}(\mu)=b_v^{(N)}. \tag{36}$$

Using (28) and (33), one obtains

$$\begin{aligned} k_v^{(N)}&=\frac{(2\mu+v-1)!!}{(2\mu-v-1)!!}b_v^{(N)}-(\Delta^{-1}f_v^{(N)}(m+1))_{m=\mu}\\ k_0^{(N)}&=2(\Delta^{-1}\gamma_2^{(N)}(m+1))_{m=\mu}. \end{aligned} \tag{37}$$

This relation gives the $k_v^{(N)}$ $(v=0, S_N+1)$ in terms of μ and of the set (b^N) of coefficients of the physical potential.

Finally from equations (23), (28), (33) and (37) one obtains the ladder and factorisation functions $K^{(N)}(x, m; \mu; b^{(N)})$ and $L^{(N)}(m; \mu; b^{(N)})$. Now, using the factorisation instruments, closed-form expressions of the perturbed eigenvalues and the normalised associated eigenfunctions of the eigenequation (34) follow at once from the results of § 2.

3.2. Perturbed eigenfunctions

Since η is the perturbation parameter, the exponential in the expression (22) of the key perturbed function ψ_{jj} can be expanded in a series of powers of η. The associated coefficients are polynomials in $\cos a(x+p)$ which can easily be obtained from the expression of $K^{(\nu)}$ (see equation (23)). It should be noted that, in a perturbation calculation to order N, the truncation up to η^N power may be done only after the normalisation (up to η^N) of the function. For the second-order perturbed key function ($N=2$), one obtains the following expression

$$\begin{aligned}\psi_{jj}=\mathscr{C}_j(\eta)\psi_{jj}^{(0)}\Bigg\{1+\sum_{v=1}^{S_2+1}\cos^v a(x+p)\\ \times\left[-\eta\frac{\varepsilon}{v}\gamma_v^{(1)}+\eta^2\left(-\frac{\varepsilon}{v}\gamma_v^{(2)}+\frac{1}{2}\sum_{t=t_0}^{t_M}\frac{\gamma_t^{(1)}\gamma_{v-t}^{(1)}}{t(v-t)}\right)\right]+\mathrm{O}(\eta^3)\Bigg\}\end{aligned} \tag{38}$$

where $\mathscr{C}_j(\eta)$ is the normalisation constant, $t_0=\max(1, v-S_1-1)$, $t_M=\min(v-1, S_1+1)$,

$$\psi_{jj}^{(0)}=C_j(j)\left[\sin\frac{a(x+p)}{2}\right]^{\varepsilon(j+d)+1/2}\left[\cos\frac{a(x+p)}{2}\right]^{\varepsilon(j-d)+1/2}, \tag{39}$$

$C_j(j)$ is the normalisation constant of the zeroth-order key function and we have used the abbreviated notation $\gamma_t^{(\nu)} = \gamma_t^{(\nu)}(m = j + \frac{1}{2}\varepsilon + \frac{1}{2};\ \mu = j;\ b^{(2)})$.

Since the perturbed factorisation scheme preserves the normalisation of the eigenfunctions, it follows that all the perturbed functions are normalised once the 'key' has been normalised.

Using the recurrence relation (7) and retaining the terms up to η^2, one obtains

$$\psi_{j,j-\varepsilon} = \psi_{jj}(2\varepsilon a^2 j)^{-1/2}\{[1 - \tfrac{1}{2}\mathcal{N}_j^{(1)} - \tfrac{1}{2}\mathcal{N}_j^{(2)} + \tfrac{3}{8}(\mathcal{N}_j^{(1)})^2](K^{(0)}(j) + K^{(0)}(j+1)) + (1 - \tfrac{1}{2}\mathcal{N}_j^{(1)})(K^{(1)}(j) + K^{(1)}(j+1)) + K^{(2)}(j) + K^{(2)}(j+1)\} \tag{40}$$

where

$$K^{(\nu)}(j) = K^{(\nu)}(x, m = j;\ \mu = j - \varepsilon;\ b^{(2)})$$
$$\mathcal{N}_j^{(\nu)} = (L^{(\nu)}(j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) - L^{(\nu)}(j + \tfrac{1}{2} - \tfrac{1}{2}\varepsilon))/(2\varepsilon a^2 j)$$

and

$$L^{(\nu)}(j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) = L^{(\nu)}(m = j + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon;\ \mu = j - \varepsilon;\ b^{(2)}).$$

When perturbed eigenfunctions ψ_{jm} far from the 'key' are required, it is advisable to use the following alternative procedure which provides the perturbed eigenfunctions in terms of the unperturbed functions. Let us set

$$\psi_{jm} = \psi_{jm}^{(0)} + \eta\psi_{jm}^{(1)} + \eta^2\psi_{jm}^{(2)} + \ldots + \eta^N\psi_{jm}^{(N)} \tag{41}$$

with

$$\psi_{jm}^{(\nu)}(x) = \sum_s \alpha_{j+s}^{(\nu)}(m)\psi_{j+s,m}^{(0)}(x) \qquad \text{and} \qquad \alpha_{j+s}^{(0)}(m) = \delta_{s,0}. \tag{42}$$

Since the perturbed functions ψ_{jm} as well as the unperturbed functions $\psi_{jm}^{(0)}$ are solutions of equation (5) with ladder functions $K(x, m)$ and $K^{(0)}(x, m)$ respectively, it follows that at each order N of the perturbation one can write

$$\sum_s \alpha_{j+s}^{(N)}(m+1)N_{j+s}^{(0)}(m+1)\psi_{j+s,m}^{(0)}(x) - \sum_{\nu=0}^{N} N_j^{(N-\nu)}(m+1)\sum_s \alpha_{j+s}^{(\nu)}(m)\psi_{j+s,m}^{(0)}(x) = -\sum_{\nu=0}^{N-1}\sum_s \alpha_{j+s}^{(\nu)}(m+1)K^{(N-\nu)}(x, m+1)\psi_{j+s,m+1}^{(0)}(x) \tag{43}$$

$$\sum_s \alpha_{j+s}^{(N)}(m)N_{j+s}^{(0)}(m+1)\psi_{j+s,m+1}^{(0)}(x) - \sum_{\nu=0}^{N} N_j^{(N-\nu)}(m+1)\sum_s \alpha_{j+s}^{(\nu)}(m+1)\psi_{j+s,m+1}^{(0)}(x) = -\sum_{\nu=0}^{N-1}\sum_s \alpha_{j+s}^{(\nu)}(m)K^{(N-\nu)}(x, m+1)\psi_{j+s,m}^{(0)}(x). \tag{44}$$

The first members of (43) and (44) involve only the functions $\psi_{j+s,m}^{(0)}$ and $\psi_{j+s,m+1}^{(0)}$ respectively; consequently it can be inferred that (42) holds if the following equations are satisfied:

$$K^{(\nu)}(x, m+1)\psi_{j,m+1}^{(0)}(x) = \sum_t A_{j+t}^{(\nu)}(j, m+1)\psi_{j+t,m}^{(0)}(x) \tag{45}$$

$$K^{(\nu)}(x, m+1)\psi_{jm}^{(0)}(x) = \sum_t B_{j+t}^{(\nu)}(j, m)\psi_{j+t,m+1}^{(0)}(x). \tag{46}$$

After substituting for $K^{(\nu)}\psi^{(0)}_{j,m+1}$ (or $K^{(\nu)}\psi^{(0)}_{jm}$) from expressions (45) (or (46)) into the equations (43) (or (44)) and equating the coefficients of the $\psi^{(0)}_{jm}$ (or of the $\psi^{(0)}_{j,m+1}$) in both sides, one finds

$$N^{(0)}_{j+s}(m+1)\alpha^{(N)}_{j+s}(m+1)-N^{(0)}_{j}(m+1)\alpha^{(N)}_{j+s}(m)$$
$$=\sum_{\nu=0}^{N-1}\Big(N_j^{(N-\nu)}(m+1)\alpha^{(\nu)}_{j+s}(m)-\sum_t A^{(N-\nu)}_{j+t}(j+s-t,\, m+1)\alpha^{(\nu)}_{j+s-t}(m+1)\Big) \tag{47}$$

$$N^{(0)}_{j+s}(m+1)\alpha^{(N)}_{j+s}(m)-N^{(0)}_{j}(m+1)\alpha^{(N)}_{j+s}(m+1)$$
$$=\sum_{\nu=0}^{N-1}\Big(N_j^{(N-\nu)}(m+1)\alpha^{(\nu)}_{j+s}(m+1)-\sum_t B^{(N-\nu)}_{j+t}(j+s-t,\, m)\alpha^{(\nu)}_{j+s-t}(m)\Big).$$

Multiplying the first equation by $N^{(0)}_j(m+1)$ and the second equation by $N^{(0)}_{j+s}(m+1)$ and adding them together, the expression for $\alpha^{(N)}_{j+s}(m)$ follows

$$\alpha^{(N)}_{j+s}(m)=C_{js}\sum_{\nu=0}^{N-1}\Big(N_j^{(N-\nu)}(m+1)(N^{(0)}_j(m+1)\alpha^{(\nu)}_{j+s}(m)+N^{(0)}_{j+s}(m+1)\alpha^{(\nu)}_{j+s}(m+1))$$
$$-\sum_t (N^{(0)}_j(m+1)A^{(N-\nu)}_{j+t}(j+s-t,\, m+1)\alpha^{(\nu)}_{j+s-t}(m+1)$$
$$+N^{(0)}_{j+s}(m+1)B^{(N-\nu)}_{j+t}(j+s-t,\, m)\alpha^{(\nu)}_{j+s-t}(m))\Big) \tag{48}$$

with $s\neq 0$ and

$$C_{js}=(L^{(0)}(j+s+\tfrac{1}{2}+\tfrac{1}{2}\varepsilon)-L^{(0)}(j+\tfrac{1}{2}+\tfrac{1}{2}\varepsilon))^{-1}.$$

For $s=0$, one can determine the $\alpha^{(N)}_j(m)$ coefficients from orthonormalisation considerations. Indeed, we must impose

$$\int_{x_1}^{x_2}\psi^*_{jm}\psi_{j'm}\,\mathrm{d}x=\delta_{jj'}. \tag{49}$$

When substituting for ψ_{jm} from (41) into the above equation, one gets the following condition to be satisfied at each order N of the perturbation

$$\sum_{\nu=0}^{N}\int_{x_1}^{x_2}\psi^{(\nu)*}_{jm}\psi^{(N-\nu)}_{j'm}\,\mathrm{d}x=0. \tag{50}$$

Since the unperturbed functions $\psi^{(0)}_{jm}$ are already orthonormalised, one gets the following expression for $\alpha^{(N)}_j(m)$

$$\alpha^{(N)}_j(m)=-\tfrac{1}{2}\sum_{\nu=1}^{N-1}\sum_s\alpha^{(\nu)}_{j+s}(m)\alpha^{(N-\nu)}_{j+s}(m). \tag{51}$$

It should be noted that $\alpha^{(1)}_j(m)=0$.

4. Illustrative application

Perturbed type A eigenequations occur in many problems of quantum mechanics and are frequently encountered in atomic and molecular physics. With the proper choice of the set of parameters involved in the type A kernel potential function, one finds the perturbed eigenequations of either the generalised spherical harmonics $Y^m_{l,\gamma}$ (which for $\gamma=\frac{1}{2}$ reduce to the usual associated spherical harmonics Y^m_l) or the symmetric top eigenfunctions $\mathscr{D}^j_{mk}$ or, more generally, the hypergeometric functions $F(\alpha,\beta;\gamma;x)$. The main results concerning the exact factorisation of the $Y^m_{l,\gamma}$, $\mathscr{D}^j_{m,k}$ and $F(\alpha,\beta;\gamma;x)$ functions are summarised in § A1.2.

Let us apply the method to the determination of perturbed type A eigenvalues and eigenfunctions up to the second order of the perturbation ($N=2$) and, in order to avoid over intricate results, let us choose the lowest values $S_1=1$ and $S_2=2S_1+1=3$ (see equation (26)).

4.1. Perturbed ladder and factorisation functions

4.1.1. First order of the perturbation N = 1. For $S_1=1$, the ladder function becomes (see equation (23))

$$K^{(1)}(x,m)=a\sin a(x+p)(\gamma_1^{(1)}(m)+\gamma_2^{(2)}(m)\cos a(x+p)) \tag{52}$$

where $\gamma_1^{(1)}$ and $\gamma_2^{(2)}$ have to be found.

From (31)

$$\gamma_2^{(1)}(m)=\frac{k_2^{(1)}}{(2m+1)(2m-1)(2m-3)}. \tag{53}$$

In order to obtain $\gamma_1^{(1)}$, one has to calculate $\Delta^{-1}f_1^{(1)}(m+1)$ which reduces to $(4d\,\Delta^{-1}\gamma_2^{(1)}(m+1))$ (see equations (28) and (29))

$$\Delta^{-1}f_1^{(1)}(m+1)=4k_2^{(1)}\Delta^{-1}\left(\frac{d}{(2m+3)(2m+1)(2m-1)}\right). \tag{54}$$

One can apply the method of decomposition into partial fractions (Jordan 1965) and introduce the 'psi' or 'digamma' function $\psi(z)=\mathrm{d}(\ln\Gamma(z))/\mathrm{d}z$ which satisfies the following functional relations (Abramowitz and Stegun 1965):

$$\begin{aligned}&\Delta\psi(z)=1/z\\&\psi(m+\tfrac{1}{2})=\psi(\tfrac{1}{2})+\sum_{k=0}^{m-1}\frac{1}{k+\frac{1}{2}}.\end{aligned} \tag{55}$$

One obtains

$$\begin{aligned}\Delta^{-1}f_1^{(1)}(m+1)&=\tfrac{1}{4}dk_2^{(1)}\Delta^{-1}\left(\frac{1}{m+\frac{3}{2}}-\frac{2}{m+\frac{1}{2}}+\frac{1}{m-\frac{1}{2}}\right)\\&=\tfrac{1}{4}d\,k_2^{(1)}\,[\psi(m+\tfrac{3}{2})-2\psi(m+\tfrac{1}{2})+\psi(m-\tfrac{1}{2})]\\&=-k_2^{(1)}\frac{d}{(2m+1)(2m-1)}.\end{aligned} \tag{56}$$

Finally

$$\gamma_1^{(1)}(m)=\frac{1}{2m(2m-2)}\left(k_1^{(1)}-k_2^{(1)}\frac{3d}{(2m+1)(2m-3)}\right). \tag{57}$$

The factorisation function $L^{(1)}(m)$ associated with $K^{(1)}(x,m)$ is given by the equation (33). It should be noted that $\Delta^{-1}\gamma_2\,(m+1)$ is already known (equation (54)).

One finds

$$L^{(1)}(m) = -a^2\left(\frac{k_2^{(1)}[m(m-1)-3d^2]}{2(2m+1)(2m-3)m(m-1)} + \frac{k_1^{(1)}d}{2m(m-1)} + k_0^{(1)}\right). \tag{58}$$

4.1.2. Second order of the perturbation $N = 2$. One has first to calculate the increments $w_v^{(2)}(m)$ arising from the first-order contribution $(K^{(1)})^2$ (see equation (30)).

The determination of the $\gamma_v^{(2)}$ ($v = 1$ to 4) of $K^{(2)}(x, m)$ (see equation (28)) is carried out in the same way as for the first order, but now one has to resort not only to the digamma function $\psi(z)$ but also to the polygamma functions $\psi^{(n)}(z)$ such that

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\psi(z) \qquad \Delta\psi^{(n)}(z) = (-1)^n n!(1/z)^{n+1}. \tag{59}$$

After some algebraic manipulations, one gets

$$\gamma_v^{(2)}(m) = \frac{(2m-v-3)!!}{(2m+v-1)!!}\left(\sum_{t=v}^{4} k_t^{(2)}P_{vt}(m) + \sum_{t=t_0}^{4} Q_{vt}(m)\sum_{r=r_0}^{r_M} k_r^{(1)}k_{t-r}^{(1)}\right) \tag{60}$$

where $t_0 = \max(2, v)$; $r_0 = \max(1, \frac{1}{2}t)$, $r_M = \min(2, t-1)$.

The expressions for the P_{ij} and Q_{ij} coefficients are reported in appendix 2.

The perturbed second-order factorisation function becomes

$$L^{(2)}(m) = -a^2((\gamma_1^{(1)}(m))^2 + 2d\gamma_1^{(2)}(m) + \gamma_2^{(2)}(m) - 2\Delta^{-1}\gamma_2^{(2)}(m+1) + k_0^{(2)}). \tag{61}$$

In fact, when performing the calculation of the $\gamma_v^{(2)}$ for $K^{(2)}(x, m)$, all terms in $L^{(2)}(m)$ have already been obtained. In particular $\Delta^{-1}\gamma_2^{(2)}$ is already known. After some rearrangements one obtains

$$L^{(2)}(m) = -a^2\left(k_0^{(2)} + \sum_{v=1}^{4} k_v^{(2)}R_v(m) + \sum_{t=2}^{4} T_t(m)\sum_{v=v_0}^{v_M} k_v^{(1)}k_{t-v}^{(1)}\right) \tag{62}$$

where $v_0 = \max(1, \frac{1}{2}t)$, $v_M = \min(2, t-1)$.

The expressions for the coefficients R_j and T_j are given in appendix 2.

4.2. Perturbed eigenvalues

One can now determine the eigenvalues of equation (34) with

$$\begin{aligned} V^{(1)}(x) &= -a^2[b_1 \cos a(x+p) + b_2 \cos^2 a(x+p)] \\ V^{(2)}(x) &= -a^2[b_3 \cos^3 a(x+p) + b_4 \cos^4 a(x+p)]. \end{aligned} \tag{63}$$

Using (37) one obtains for the first order of the perturbation ($N = 1$)

$$k_2^{(1)} = (4\mu^2 - 1)b_2 \qquad k_1^{(1)} = 2\mu b_1 + d\, b_2 \qquad k_0^{(1)} = -\tfrac{1}{2}b_2 \tag{64}$$

and for the second order of the perturbation ($N = 2$)

$$\begin{aligned} k_4^{(2)} &= (4\mu^2-1)(4\mu^2-9)b_4 - 2(4\mu^2-1)b_2^2 \\ k_3^{(2)} &= \tfrac{3}{2}(8\mu^2-3)db_4 + 8\mu(\mu^2-1)b_3 - 6\mu b_1 b_2 \\ k_2^{(2)} &= [\tfrac{3}{2}(4\mu^2-1)+d^2]b_4 + 4\mu d b_3 - \tfrac{1}{4}b_2^2 - b_1^2 \\ k_1^{(2)} &= \tfrac{5}{2}db_4 + 2\mu b_3 \qquad k_0^{(2)} = -\tfrac{3}{8}b_4. \end{aligned} \tag{65}$$

The perturbed eigenvalues follow from (62), (64), (65) and (21):

$$\Lambda_{jm} = a^2(\Lambda_j^{(0)} + \Lambda_{jm}^{(1)} + \Lambda_{jm}^{(2)}) \tag{66}$$

with

$$\Lambda_j^{(0)} = (j + \tfrac{1}{2}\varepsilon)^2$$

$$\Lambda_{jm}^{(1)} = -4b_1 \frac{md}{J} + \tfrac{1}{2} b_2\left(1 + \frac{J(1-4m^2-4d^2)+48m^2d^2}{J(J-3)}\right)$$

$$\begin{aligned}\Lambda_{jm}^{(2)} = {} & \frac{2b_3 md}{J(J-3)(J-8)}[-3J^2+18J-32+4(3J-4)(m^2+d^2)-80m^2d^2] \\ & + \frac{b_4}{8J(J-3)(J-8)(J-15)}\{3J(J-8)[J^2-16J+24+8(10-J)(m^2+d^2) \\ & + 16(m^4+d^4)] + 64m^2d^2[9J^2-102J \\ & + 380 + 10(10-3J)(m^2+d^2) + 140m^2d^2]\} \\ & + \frac{b_1^2}{2J^3(J-3)}[J^3 - 12J^2(m^2+d^2) + 16(5J+12)m^2d^2] \\ & + \frac{8b_1b_2md}{J^3(J-3)(J-8)}[J^2(4-3J)+20J^2(m^2+d^2)-16(7J+24)m^2d^2] \\ & + \frac{b_2^2}{8J^3(J-3)^3(J-15)}\{J^6 - 15J^5 + 68J^4 - 48J^3 \\ & - 8J^3(3J^2-13J+60)(m^2+d^2) + 16J^3(5J+33)(m^4+d^4) \\ & + 64m^2d^2[J^2(25J^2-87J+180) + 18J^2(5-7J)(m^2+d^2) \\ & + 36(17J^2+57J-180)m^2d^2]\}\end{aligned}$$

where $J = 4(j + \frac{1}{2}\varepsilon + \frac{1}{2})(j + \frac{1}{2}\varepsilon - \frac{1}{2})$.

As expected the expressions of the $\Lambda_{jm}^{(N)}$ are symmetric functions of m and d (see appendix 1) and depend on j via $J = 4j(j+1)$ for class I problems (or $J = 4j(j-1)$ for class II problems).

For the case of the perturbed spherical harmonics ($d = 0$, $l = 1$, $\varepsilon = +1$), the expression of Λ_{jm} reduces considerably and is amenable, after change of notation, to known formulae (for $b_1 = b_3 = b_4 = 0$, see for instance, Abramowitz and Stegun (1965); for $b_2 = b_3 = b_4 = 0$ see Winjberg (1974)).

4.3. *Perturbed eigenfunctions*

In order to avoid rather long expressions, we limit ourselves to the first order of the perturbation with $S_1 = 1$. As an example, let us consider the determination of the class I ($\varepsilon = +1$; $0 \leq a(x+p) \leq \pi$) eigenfunctions of the eigenequation (34) with the potential $V^{(1)}$ (equation (63)).

The ladder function is

$$K(x, m) = am \cot a(x+p) + \frac{ad}{\sin a(x+p)} + a \sin a(x+p)[\gamma_1^{(1)} + \gamma_2^{(1)} \cos a(x+p)] \tag{67}$$

where

$$\gamma_1^{(1)}=\frac{\mu b_1}{2m(m-1)}+\frac{[m(m-1)-3\mu^2]db_2}{m(m-1)(2m+1)(2m-3)}\qquad \gamma_2^{(1)}=\frac{(4\mu^2-1)b_2}{(2m-3)(4m^2-1)}.$$

The normalised key function is (see equation (38))

$$\psi_{jj}=\psi_{jj}^{(0)}\bigg(1-\frac{db_1}{2(j+1)^2}+\frac{b_2[(j+1)^2+2d^2(3j+4)]}{2(j+1)^2(2j+3)^2}$$
$$+\frac{1}{(j+1)}\left(-\frac{b_1}{2}+\frac{b_2d}{(2j+3)}\right)\cos a(x+p)-\frac{1}{2}\frac{b_2}{(2j+3)}\cos^2 a(x+p)\bigg) \qquad (68)$$

where the unperturbed key function $\psi_{jj}^{(0)}$ is given by (39) with

$$C_j(j)=\left(\frac{(2j+1)!}{\Gamma(j+d+1)\Gamma(j-d+1)}\right)^{1/2}.$$

Starting from ψ_{jj}, one can generate, stepwise, the whole set of the ψ_{jm} eigenfunctions using either the difference differential equations (5) or the recurrence relation (7).

In order to obtain closed-form expressions of the ψ_{jm} eigenfunctions in terms of the unperturbed $\psi_{jm}^{(0)}$ functions, the critical step of the calculation is to expand the term $K^{(1)}\psi_{jm}^{(0)}$, i.e. $\sin a(x+p)\psi_{jm}^{(0)}$ and $\cos a(x+p)\psi_{jm}^{(0)}$, in the basis of the $\psi_{jm}^{(0)}$. Due to the correspondence between type A and type E eigenequations which interchanges the roles of the quantum numbers j and m, the expansions required follow from the recurrence relation (7) for types A and E (see equations (A7)).

Using the results of § 3, one obtains

$$\psi_{jm}=\sum_{s=-2}^{2}\alpha_{j+s}^{(1)}(m)\psi_{j+s,m}^{(0)} \qquad (69)$$

where

$$\alpha_j^{(1)}(m)=1$$

$$\alpha_{j+2}^{(1)}(m)=\frac{-b_2\{[(j+2)^2-d^2][(j+1)^2-d^2][(j+2)^2-m^2][(j+1)^2-m^2]\}^{1/2}}{2(2j+3)^2(j+2)(j+1)[(2j+5)(2j+1)]^{1/2}}$$

$$\alpha_{j+1}^{(1)}(m)=\frac{[mdb_2-\frac{1}{2}b_1j(j+2)]\{[(j+1)^2-d^2][(j+1)^2-m^2]\}^{1/2}}{j(j+2)(j+1)^2[(2j+3)(2j+1)]^{1/2}}$$

$$\alpha_{j-1}^{(1)}(m)=\frac{[\frac{1}{2}b_1(j^2-1)-mdb_2][(j^2-d^2)(j^2-m^2)]^{1/2}}{(j^2-1)j^2[(2j+1)(2j-1)]^{1/2}}$$

$$\alpha_{j-2}^{(1)}(m)=\frac{b_2\{(j^2-d^2)[(j-1)^2-d^2](j^2-m^2)[(j-1)^2-m^2]\}^{1/2}}{2(2j-1)^2j(j-1)[(2j+1)(2j-3)]^{1/2}}.$$

It should be noted that, as expected, the $\alpha_{j+s}^{(1)}(m)$ coefficients are symmetric functions of m and d. Let us point out that expression (69) when $d=0$ and $b_2=0$ reduces considerably and gives again known results (see, for instance, Kusch and Hughes 1969).

5. Conclusion

Finally, if we compare this work with papers I and II, we can say that roughly they involve dual points of view of the perturbed factorisation treatment. Indeed, in papers I and II, when building the factorisation instruments, we have adopted the Infeld–Hull suggestion and assumed for these functions the traditional polynomial m dependence. Then the x dependences of the ladder function $K(x, m)$ and the factorising potential $U(x, m)$ follow from the factorisability condition. Although this point of view was suitable for treating the radial cases (types B, C, D and F), when considering the trigonometric cases (types A and E) we obtained very intricate expressions for the factorising potential function $U(x, m)$. In the present paper, it is the x dependence of the functions which is treated first, then the m dependence is obtained from the solution of the factorisability condition. It should be noted that this way of proceeding is not merely a matter of computational convenience; in fact the use of finite difference calculus enables one to overstep the narrow limits of the usual polynomial dependence in the quantum number m of the perturbed ladder operators. Let us mention that this procedure, which can be applied without any restriction to all factorisable kernel types, allows one to recognise at once if a given eigenequation involving a physico–mathematical potential is amenable to perturbed factorisation.

An additional advantage of the present method is that the rather tedious part of the calculation carried on in paper I, which ensures that the given potential $V(x, m)$ matches the factorising potential $U(x, m)$, is no longer necessary.

As a by-product of the computation of the perturbed functions, it has been shown how the existence of an interrelation between factorisable type A and type E eigenequations leads to a family of recurrence relations. These relations are of particular interest for analytical computation of matrix elements. Results concerning all types of factorisation will be given elsewhere.

Appendix 1

A1.1. Recurrence relations for type A eigenfunctions

Let us consider the type A eigenequation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\frac{a^2(m^2+d^2-\frac{1}{4}+2dm\cos a(x+p))}{\sin^2 a(x+p)}+a^2(j+\tfrac{1}{2}\varepsilon)^2\right)\psi_{jm}^{(0)}(x)=0 \tag{A1}$$

and let us set

$$az=\ln\tan\tfrac{1}{2}a(x+p) \qquad \phi(z)=g_j^{-1}(\sin a(x+p))^{-1/2}\psi_{jm}^{(0)}(x) \tag{A2}$$

where g_j^{-1} is a normalising factor. Following the argument given in § 9-1 of Infeld and Hull (1951), it is easily seen that g_j does not depend on m. Then the type A eigenequation (A1) becomes a type E eigenequation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\frac{A^2l(l+1)}{\sin^2 A(z+P)}-2Aq(m)\cot A(z+P)+A^2(m^2+d^2)\right)\phi_{ml}^{(0)}(z)=0 \tag{A3}$$

where $A=-\mathrm{i}a$; $P=\mathrm{i}\pi/2a$; $l=j+\frac{1}{2}\varepsilon-\frac{1}{2}$; $q(m)=-adm$ and $\varepsilon=+1$ (or $\varepsilon=-1$) according to the class I (or class II) for type A.

The type E eigenfunctions $\phi_{ml}^{(0)}(z)$ are solutions of the pair of difference differential equations (5) with the associated ladder function which is defined in table 1. Using the changes of variable and function defined by equation (A2), one obtains

$$\left(-a(l+\tfrac{1}{2})\cos a(x+p)+\frac{q(m)}{l}+\sin a(x+p)\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm}^{(0)}=\frac{g_j}{g_{j-1}}N'_j(m)\psi_{j-1,m}^{(0)}$$

$$\left(-a(l+\tfrac{1}{2})\cos a(x+p)+\frac{q(m)}{l+1}-\sin a(x+p)\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm}^{(0)}=\frac{g_j}{g_{j+1}}N'_{j+1}(m)\psi_{j+1,m}^{(0)} \qquad \text{(A4)}$$

where

$$N'_j(m)=\frac{1}{l}[a^2(l^2-m^2)(l^2-d^2)]^{1/2}$$

$$g_{j+1}/g_j=-(C_{j+1}(j+1)/C_j(j))\left(\frac{(l+1)^2-d^2}{2(2j+1)(l+1)}\right)^{\varepsilon/2}.$$

$C_j(m)$ is the normalisation constant for type A $\psi_{jm}^{(0)}$ eigenfunctions (see table 1). In particular, for class I problems ($\varepsilon=+1$)

$$C_j(m)=\left(\frac{(j-m)!(2j+1)\Gamma(j+m+1)}{\Gamma(j+d+1)\Gamma(j-d+1)}\right)^{1/2}. \qquad \text{(A5)}$$

These equations (A4) generate eigenfunctions $\psi_{jm}^{(0)}$ with a given value of m step by step, downward or upward, and allow the determination of any solution $\psi_{j+t,m}^{(0)}$ from the knowledge of $\psi_{jm}^{(0)}$.

On the other hand, since the $\psi_{jm}^{(0)}$ are solutions of a type A factorisable equation (A1), they are also solutions of the following difference differential equations (see equation (5) and table 1 for type A):

$$\left(a(m-\tfrac{1}{2})\cot a(x+p)+\frac{ad}{\sin a(x+p)}+\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm}^{(0)}=N_j(m)\psi_{j,m-1}^{(0)}$$

$$\left(a(m+\tfrac{1}{2})\cot a(x+p)+\frac{ad}{\sin a(x+p)}-\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_{jm}^{(0)}=N_j(m+1)\psi_{j,m+1}^{(0)} \qquad \text{(A6)}$$

where

$$N_j(m)=[a^2(j+\tfrac{1}{2}\varepsilon-\tfrac{1}{2}+m)(j+\tfrac{1}{2}+\tfrac{1}{2}\varepsilon-m)]^{1/2}.$$

Using both the equations (A4) and (A6), one finds the following recurrence relations

$$\cos a(x+p)\psi_{jm}^{(0)}=A_j[(l+1)^2-m^2]^{1/2}\psi_{j+1,m}^{(0)}-\frac{md}{l(l+1)}\psi_{jm}^{(0)}+B_j(l^2-m^2)^{1/2}\psi_{j-1,m}^{(0)} \qquad \text{(A7)}$$

$$\begin{aligned}\sin a(x+p)\psi_{jm}^{(0)}&\\ &=A_j[(l+m+2)(l+m+1)]^{1/2}\psi_{j+1,m+1}^{(0)}\\ &\quad+\frac{d[(l+m+1)(l-m)]^{1/2}}{l(l+1)}\psi_{j,m+1}^{(0)}-B_j[(l-m)(l-m-1)]^{1/2}\psi_{j-1,m+1}^{(0)}\end{aligned}$$

$$\begin{aligned}\sin a(x+p)\psi_{jm}^{(0)}&\\ &=-A_j[(l-m+2)(l-m+1)]^{1/2}\psi_{j+1,m-1}^{(0)}+\frac{d[(l+m)(l-m+1)]^{1/2}}{l(l+1)}\psi_{j,m-1}^{(0)}\\ &\quad+B_j[(l+m)(l+m-1)]^{1/2}\psi_{j-1,m-1}^{(0)}\end{aligned}$$

with

$$A_j = \frac{1}{(l+1)(2l+1)} \frac{C_j(j)}{C_{j+1}(j+1)} \left(\frac{2(2j+1)(l+1)}{(l+1)^2 - d^2}\right)^{\varepsilon/2} [(l+1)^2 - d^2]^{1/2}$$

$$B_j = \frac{1}{l(2l+1)} \frac{C_j(j)}{C_{j-1}(j-1)} \left(\frac{l^2 - d^2}{2l(2j-1)}\right)^{\varepsilon/2} (l^2 - d^2)^{1/2}.$$

It should be noted that the arbitrary constants a, d and p may be real or complex. The associated range of x has to be such that the ladder function $K(x, m)$ has no singularities more severe than first order. Thus, setting $a(x+p) = x + \mathrm{i}y$, allowed ranges for x are defined by the following conditions

$$\begin{array}{ll} 0 < x < \pi & \text{if } y = 0 \\ 0 < y < \infty & \text{if } y \neq 0 \text{ and } x = 0 \\ -\infty < y < \infty & \text{if } y \neq 0 \text{ and } x = \pm\frac{1}{2}(2n+1)\pi. \end{array}$$

A1.2. Some particular type A eigenfunctions

A1.2.1. Associated spherical harmonics.

$$Y_l^m(\theta, \phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{im\phi} (\sin\theta)^{-1/2} \psi_{lm}(\theta)$$

$\psi_{lm}(\theta)$ is a class I ($\varepsilon = +1$) solution of (A1) with $x = \theta$; $0 \leqslant \theta \leqslant \pi$; $a = 1$; $d = p = 0$ and $j - m = l - m =$ positive integer or zero.

A1.2.2. Generalised spherical harmonics.

$$Y_{l,\gamma}^m(\theta, \phi) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{im\phi} (\sin\theta)^{-\gamma} \psi_{l+\gamma-1/2, m+\gamma-1/2}(\theta)$$

$\psi(\theta)$ is a class I ($\varepsilon = +1$) solution of (A1) with $x = \theta$; $0 \leqslant \theta \leqslant \pi$; $a = 1$; $d = p = 0$ and $j \to l + \gamma - 1/2$; $m \to m + \gamma - 1/2$; $j - m = l - m =$ positive integer or zero. When $\gamma = \frac{1}{2}$, one again obtains the usual spherical harmonics Y_l^m.

A1.2.3. Symmetric top functions.

$$\mathscr{D}_{mk}^j(\alpha, \beta, \gamma) = \mathrm{e}^{im\alpha} d_{mk}^j(\beta) \, \mathrm{e}^{ik\gamma}$$

where α, β, γ are the three Euler angles

$$0 \leqslant \alpha \leqslant 2\pi \qquad 0 \leqslant \beta \leqslant \pi \qquad 0 \leqslant \gamma \leqslant 2\pi$$

$$\frac{1}{8\pi^2} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \sin\beta \, \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\gamma \, \mathscr{D}_{mk}^{j*} \mathscr{D}_{mk}^j = \frac{1}{2j+1}$$

$$d_{mk}^j(\beta) = \left(\frac{2}{2j+1}\right)^{1/2} (\sin\beta)^{-1/2} \psi_{jm}(\beta)$$

$\psi_{jm}(\beta)$ is a class I ($\varepsilon = +1$) solution of (A1) with $x = \beta$; $a = 1$; $p = 0$; $d = k$.

Since both differences $j - m$ and $j - k$ are positive integers or zero, dual factorisation of equation (A1) is possible with quantum number either m or $k = d$.

A1.2.4. Hypergeometric functions. The differential equation satisfied by the hypergeometric function $F(\alpha, \beta; \gamma; x)$ is (Gradshteyn and Ryzhik 1980)

$$\left(x(1-x)\frac{d^2}{dx^2}+[\gamma-(\alpha+\beta+1)x]\frac{d}{dx}-\alpha\beta\right)F(x)=0. \tag{A8}$$

Setting $x=\sin^2 y$ and $F=(\sin y)^{-\gamma+1/2}(\cos y)^{-\alpha-\beta+\gamma-1/2}\psi(y)$, one obtains

$$\left(\frac{d^2}{dy^2}-\frac{(\gamma-\frac{3}{2})(\gamma-\frac{1}{2})}{\sin^2 y}-\frac{(\alpha+\beta-\gamma-\frac{1}{2})(\alpha+\beta-\gamma+\frac{1}{2})}{\cos^2 y}+(\alpha-\beta)^2\right)\psi(y)=0.$$

It is easily seen that equation (A8) can be rewritten

$$\left(\frac{d^2}{dy^2}-\frac{4}{\sin^2 2y}(m^2+d^2-\tfrac{1}{4}+2md\cos 2y)+(\alpha-\beta)^2\right)\psi(y)=0 \tag{A9}$$

with

$$m=\tfrac{1}{2}(\alpha+\beta-1) \qquad d=\tfrac{1}{2}(2\gamma-\alpha-\beta-1).$$

This is a factorisable type A (Class I) eigenequation with $a=2$; $p=0$. The quantisation condition requires $j-m=v=$ positive integer or zero. The eigenvalue is

$$\lambda=L(j+1)=L(m+v+1)=4\left(\frac{\alpha+\beta}{2}+v\right)^2. \tag{A10}$$

On the other hand, the eigenvalue in (A9) is seen to be

$$\lambda=(\alpha-\beta)^2. \tag{A11}$$

From equations (A10) and (A11) either $v=-\alpha$ or $v=-\beta$. We thus rediscover the well known condition for finite hypergeometric series: at least one of α, β is a non-positive integer.

Appendix 2

A2.1. Expressions for the coefficients $P_{ij}(m)$ and $Q_{ij}(m)$ of the perturbed second-order ladder function $K^{(2)}(x, m)$

$\lambda=4m(m-1)$.

$$P_{44}=1 \qquad P_{33}=1 \qquad P_{34}=\frac{-5d(2\lambda-9)}{2(\lambda-3)(\lambda-15)}$$

$$P_{22}=1 \qquad P_{23}=\frac{-2d(2\lambda-1)}{\lambda(\lambda-8)} \qquad P_{24}=\frac{3[10d^2(\lambda-1)-\lambda(\lambda-8)]}{2\lambda(\lambda-8)(\lambda-15)}$$

$$P_{11}=1 \qquad P_{12}=\frac{-3d}{\lambda-3} \qquad P_{13}=\frac{10d^2-\lambda+3}{(\lambda-3)(\lambda-8)}$$

$$P_{14}=\frac{5d(5\lambda-26-28d^2)}{4(\lambda-3)(\lambda-8)(\lambda-15)}.$$

$$Q_{44}=\frac{3}{\lambda-3} \qquad Q_{33}=\frac{5}{\lambda-3} \qquad Q_{34}=\frac{-3d(7\lambda+15)(\lambda-8)}{(\lambda-3)^2(\lambda-15)(\lambda+1)}$$

$$Q_{22}=\frac{2\lambda+3}{\lambda^2}\qquad Q_{23}=\frac{-24d(\lambda+2)}{\lambda^2(\lambda-8)}$$

$$Q_{24}=\frac{4d^2(68\lambda^3+159\lambda^2-378\lambda-405)-\lambda^2(15\lambda^2+2\lambda+51)}{4\lambda^2(\lambda-3)^2(\lambda-15)(\lambda+1)}$$

$$Q_{12}=\frac{-d(5\lambda+12)}{\lambda^2(\lambda-3)}\qquad Q_{13}=\frac{8d^2(7\lambda+24)-5\lambda^2}{\lambda^2(\lambda-3)(\lambda-8)}$$

$$Q_{14}=\frac{d[-36d^2(17\lambda^2+57\lambda-180)(\lambda+1)+\lambda^2(91\lambda^2-6\lambda-225)]}{4\lambda^2(\lambda+1)(\lambda-3)^3(\lambda-15)}.$$

A2.2. Expressions for the $R_j(m)$ and $T_j(m)$ coefficients of $L^{(2)}(m)$

$$\lambda=4m(m-1).$$

$$R_4=-\frac{\frac{3}{8}\lambda(\lambda-8)+5d^2(10-3\lambda)+70d^4}{\lambda(\lambda-3)(\lambda-8)(\lambda-15)}$$

$$R_3=-\frac{d(3\lambda-4-20d^2)}{\lambda(\lambda-3)(\lambda-8)}\qquad R_2=-\frac{12d^2-\lambda}{2\lambda(\lambda-3)}\qquad R_1=\frac{2d}{\lambda}.$$

$$T_4=-\frac{\frac{1}{8}\lambda^3(5\lambda+33)+9d^2\lambda^2(5-7\lambda)+18d^4(17\lambda^2+57\lambda-180)}{\lambda^3(\lambda-3)^3(\lambda-15)}$$

$$T_3=-\frac{4d[5\lambda^2-4d^2(7\lambda+24)]}{\lambda^3(\lambda-3)(\lambda-8)}\qquad T_2=-\frac{4d^2(5\lambda+12)-3\lambda^2}{2\lambda^3(\lambda-3)}.$$

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